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Euler–Poincaré flows and leibniz structure of nonlinear reaction–diffusion type systems

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Abstract

In this paper we present Euler–Poincaré formulation of the Fisher, Fitzhugh–Nagumo, Burgers–Huxley and extended Fitzhugh–Nagumo and extended Burgers–Huxley type nonlinear reaction–diffusion systems. All these flows are related to infinite dimensional almost Poisson manifolds and the corresponding Lie–Poisson structures yield Leibniz brackets, a bracket endowed with both symmetric and skewsymmetric parts. The symmetric part contributes the diffusion part of the ssystem. The properties exhibited by the reaction–diffusion systems defined in this way are in general very different from the standard Hamiltonian mechanics since the dynamics are controlled by the standard Poisson brackets. Moreover, all the nonlinear reaction–diffusion systems under consideration are Euler–Poincaré flows on the dual of Kirillov's superalgebra associated to the Bott–Virasoro group.

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1. Introduction

Nonlinear differential equations are known to describe a wide variety of physical phenomena [1,2,7]. These equations model diverse real-life phenomena in physics, chemistry, biology, physiology and other fields. Nonlinear reaction–diffusion systems are known to exhibit many

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interesting spatio-temporal patterns. Fisher equation is a prototype of diffusive equations, given as

$$u_t - u_{xx} + u(1-u) = 0. \tag{1}$$

Numerous physical phenomena, such as wall propagation in liquid crystals, nerve pulse propagation in nerve fibres, are closely connected with the study of nonlinear diffusion equations of reactive type. These equations play an important role in dissipative dynamical systems.

The Hodgkin–Huxley model was developed to characterize the action potential of a squid axon. It has served as an archetype for compartmental models of the electrophysiology of biological membranes [12]. The Fitzhugh–Nagumo nerve conduction equation [7] is the simplest version of the above model and it is given by

$$u_t - u_{xx} + u(u-1)(u-\gamma) = 0 \tag{2}$$

does not possess the Painlevé property but it has the conditional Painlevé property instead, that is, the Painlevé property is enjoyed just by a subset of the solutions of the PDE under study [6]. Travelling wave solutions of the Fitzhugh–Nagumo equation are known to us [1,2,20]. Exact solutions of this equation have been obtained using various techniques [cf. 19 and references therein]. In this paper we will study the Euler–Poincaré formalism of the Fitzhugh–Nagumo type equation. In particular, we will study EPDiff forms of nonlinear reaction–diffusion systems. EPDiff [cf. 11]is a short form of Euler–Poincaré equation on diffeomorphism.

We will also consider a generalization of the Fitzhugh–Nagumo equation, known as the Burgers–Huxley equation [cf. 6]from which the former is a particular case of the other. The Burgers–Huxley equation is

$$u_t - u_{xx} + \alpha u u_x + \beta u(u-1)(u-\gamma) = 0. \tag{3}$$

Most recently, Kyrychko et al. [15] studied the extended Burgers–Huxley equation with the fourth-order derivative:

$$u_t = -\delta u_{xxxx} + u_{xx} - \alpha u u_x - \beta u(u-1)(u-\gamma) \tag{4}$$

with parameters $\alpha > 0$, $\beta > 0$, $\gamma < 0$, $\delta > 0$, where the fourth-order derivative term is added to account for long-range effects.

The Lie–Poisson type brackets appear in this paper possess both symmetric and antisymmetric structure. The properties of the Poisson bracket have important consequences on the dynamical features of the Hamiltonian vector field $X_H$in ordinary mechanics, for example, the skewsymmetric condition implies that the Hamiltonian function H is a constant of the motion for X_H.

It has been noticed recently that a different type of Poisson bracket is sometimes necessary to incorporate dissipative type systems. A well known example is almost Poisson brackets, the brackets do not satisfy Jacobi identity, are employed to study non-holonomic constrained system. Morrison [19] and Brockett [see for example, 4]have proposed the modeling of certain dissipative phenomena by adding a symmetric bracket to a known antisymmetric one. This new bracket is called Leibniz bracket, given as

$$[\cdot,\cdot]_{\text{Leibniz}} = \{\cdot,\cdot\}_{\text{skew}} + \{\{\cdot,\cdot\}\}_{\text{sym}},$$

where the bracket $\{\cdot,\cdot\}_{\text{skew}}$, the skewsymmetric, $\{\{\cdot,\cdot\}\}_{\text{sym}}$, the symmetric, and the sum is a Leibniz bracket. This bracket captures the modeling of a surprising number of physical examples [3,8–10][3,8–10].

An **example** that fits into Leibniz mechanics framework [3] is the Landau-Lifschitz equation for the magnetization vector $\mathbf{m}$in an external vector field $\mathbf{B}$:

$$\dot{\mathbf{m}} = \gamma\mathbf{m} \times \mathbf{B} + \frac{\beta}{||\mathbf{m}||^2}(\mathbf{m} \times (\mathbf{m} \times \mathbf{B})), \tag{5}$$

where $\gamma$and $\beta$are physical parameters. The Leibniz bracket on $\mathbf{R}^3$associated to the Landau-Lifschitz equation given by the sum of the two brackets:

$$\{f, g\}_{\text{skew}}(\mathbf{m}) := \mathbf{m}(\nabla f(\mathbf{m}) \times \nabla g(\mathbf{m})),$$
$$\{\{f, g\}\}_{\text{sym}}(\mathbf{m}) := \frac{\beta(\mathbf{m} \times \nabla f(\mathbf{m}))(\mathbf{m} \times \nabla g(\mathbf{m}))}{\gamma||\mathbf{m}||^2}.$$

In this paper we will consider a field theoretic analog of symmetric and Leibniz type brackets to describe the reaction–diffusion type systems. These Lie–Poisson or Leibniz–Poisson brackets consist of two parts: symmetric and skew symmetric parts. The diffusion term appears from the symmetric part of the Leibniz bracket. We show that the Fisher, the Burgers–Huxley, etc. type systems can be described using this mathematical construction that generalizes the standard Poisson bracket currently used in Hamiltonian mechanics [17].

Organization: The plan of the paper is as follows. In Section 2, we introduce the Virasoro algebra and Kirillov's construction of local or superalgebra. We also briefly review Lie–Poisson structure. In Section 3 we study Euler–Poincaré flows on dual of Kirillov's superalgebra and its relation to the Fisher and certain reduction of the Fitzhugh–Nagumo equation. In Section 4, we discuss Euler–Poincaré forms of the Fitzhugh–Nagumo equation and Section 5 is devoted to the Burgers–Huxley type systems. All these systems are connected to Leibniz–Poisson bracket. Finally, in Section 6, we make a number of final comments and briefly outline a possible connection of the present work to the Lie pseudogroup and confomal Hamiltonian systems [5,18,21,22,25].

1.1. Result of the paper

Our main result is as follows:

Theorem 1.1. *Let G be a superconformal group equipped with a metric* $\langle\cdot, \cdot\rangle$*which is invariant under right translation and the Lie algebra associated to G is given by a pair* $\left(\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a\right)\left(\phi(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}, b\right)\right)$, *where* $a, b \in \mathbf{R}$*denote the central extension. The Euler–Poincaré flow on the odd part* $(f = 0)$*of the dual space of superconformal algebra yields the Fisher equation and the dispersionless Fitzhugh–Nagumo equation* (*associated to the hyperplane* $b = 0$).

Corollary 1.2. *The Lie–Poisson bracket associated to the Fisher equation*:

$$\{\{f, g\}\}_{\text{sym}} := \int_{S^1} \frac{\delta f}{\delta u}\mathcal{O}_{\text{Fisher}}\frac{\delta g}{\delta u}, \qquad \mathcal{O}_{\text{Fisher}} = \left(-c\frac{d^2}{dx^2} + u(x) + \Lambda\right) \tag{6}$$

is a symmetric bracket and does not satisfy usual skew symmetric properties.

Theorem 1.3. *Let* $\mathcal{O}_{BH} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x + \beta(u + \Lambda)$*be the one parameter Hamiltonian operator associated to the flow on the odd part* $(f = 0)$*of the superconformal group and the space of* *first-order differential operators. The Euler–Poincaré flow* $u_t = \mathcal{O}_{BH}\frac{\delta H}{\delta u}$ *on this combined space yields the Burgers–Huxley type equations for* $H = \frac{1}{2}\int_{S^1} u^2\,\mathrm{d}x$.

Corollary 1.4. *The Lie–Poisson bracket associated to the Burgers–Huxley equation*:

$$\{f, g\}_{\text{Leib}} := \int_{S^1} \frac{\delta f}{\delta u}\mathcal{O}_{BH}\frac{\delta g}{\delta u} \tag{7}$$

is a Leibniz bracket, it contains both symmetric and skew symmetric structure.

2. Euler–Poincaré flows and Kirillov superalgebra

Let $Diff(S^1)$be the group of orientation preserving diffeomorphisms of the circle. It is known that the group $Diff(S^1)$as well as its Lie algebra of vector fields on S^1, $T_{id}Diff(S^1) = Vect(S^1)$, have non-trivial one-dimensional central extensions, the Bott–Virasoro group $\hat{D}iff(S^1)$and the Virasoro algebra *Vir*, respectively.

The Lie algebra $Vect(S^1)$is the algebra of smooth vector fields on S^1. This satisfies the commutation relations:

$$\left[f\frac{\mathrm{d}}{\mathrm{d}x}, g\frac{\mathrm{d}}{\mathrm{d}x}\right] := (f(x)g'(x) - f'(x)g(x))\frac{\mathrm{d}}{\mathrm{d}x}. \tag{8}$$

Let us consider the Lie algebra of vector fields on S^1, $Vect(S^1)$. The dual of this algebra is identified with space of quadratic differential forms $u(x)\mathrm{d}x^{\otimes 2}$by the following pairing:

$$\langle u(x), f(x)\rangle = \int_0^{2\pi} u(x)\, f(x)\mathrm{d}x,$$

where $f(x)\frac{\mathrm{d}}{\mathrm{d}x} \in Vect(S^1)$.

2.1. Virasoro action and Virasoro orbit

The algebra of vector field $Vect(S^1)$has a unique non-trivial central extension by means of $\mathbf{R}$:

$$0 \longrightarrow \mathbf{R} \longrightarrow \text{Vir} \longrightarrow Vect(S^1)$$

described by the Gelfand–Fuchs cocycle $\omega_1(f, g) = \frac{1}{2}\int_{S^1} f'g''\,\mathrm{d}x$, and it is called the Virasoro algebra *Vir*.

The elements of *Vir* can be identified with the pairs (2πperiodic function, real number). The commutator in *Vir* takes the form:

$$\left[\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right] = \left((fg' - gf')\frac{\mathrm{d}}{\mathrm{d}x}, \int_{S^1}\frac{1}{2}f'g''\,\mathrm{d}x\right).$$

The dual space Vir^*can be identified to the set $(c, u\,\mathrm{d}x^2)|c \in \mathbf{R}$.

A pairing between a point $(\lambda, f(x)\frac{\mathrm{d}}{\mathrm{d}x}) \in$ Virand a point $(c, u\,\mathrm{d}x^2)$is given by

$$ac + \int_{S^1} f(x)u(x)\mathrm{d}x.$$

One parameter family of $Vect(S^1)$acts on the space of smooth functions $C^\infty(S^1)$by

$$\mathcal{L}^{(\mu)}_{f(x)\mathrm{d}/\mathrm{d}x}p(x) = f(x)p'(x) - \mu f'(x)p(x), \tag{9}$$

where

$$\mathcal{L}^{(\mu)}_{f(x)\mathrm{d}/\mathrm{d}x} = f(x)\frac{\mathrm{d}}{\mathrm{d}x} - \mu f'(x) \tag{10}$$

is the derivative with respect to the vector field $f(x)\frac{\mathrm{d}}{\mathrm{d}x}$and $p(x) \in C^{\infty}(S^1)$. Eq. (5) implies a one parameter family of $Vect(S^1)$action on the space of smooth functions $C^{\infty}(S^1)$.

Let us denote $\mathcal{F}_{\mu}(M)$the space of tensor densities of degree $-\mu$:

$$\mathcal{F}_{\lambda} = p(x)\mathrm{d}x^{-\lambda} | p(x) \in C^{\infty}(S^1).$$

Thus, we say

$$\mathcal{F}_{-\lambda} \in \Gamma(\Omega^{\otimes\lambda}), \qquad \Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda},$$

where $\mathcal{F}_0(M) = C^{\infty}(M)$, the space $\mathcal{F}_{-1}(M)$coincides with the space differential forms.

Hence, the equation can be interpretred as an action of $Vect(S^1)$on $\mathcal{F}_{\mu}(S^1)$, a tensor densities on $S^1$of degree μ.

The *regular part* of the dual of the Virasoro algebra is $C^{\infty}(S^1) \oplus \mathbf{R}$, and a pairing between this space and Virasoro algebra is given by

$$\langle (u(x), c), \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a \right) \rangle := \int_{S^1} u(x) f(x) \mathrm{d}x + ca.$$

Using the following equation:

$$\langle ad^*_{(f(x)\mathrm{d}/\mathrm{d}x, a)}(u(x), c), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b \right) \rangle = \langle (u(x), c), ad_{(f(x)\mathrm{d}/\mathrm{d}x, a)} \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b \right) \rangle,$$

we obtain

$$\tilde{u} = \tfrac{1}{2} f''' + 2f'u + fu' \Longrightarrow \left(\tfrac{1}{2}\partial_x^3 + 2u\partial_x + u_x \right) f.$$

The operator $\left(\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x \right)$is called the implectic or Poisson operator of the KdV equation.

2.1.1. Historical remark

If a nonlinear evolution equation is written as $u_t = K(u)$, can be represented in the following way:

$$u_t = K(u) = \Lambda_1^{\sharp}(u) f_1(u) = \Lambda_2^{\sharp}(u) f_2(u),$$

where $f_1(u)$, $f_2(u)$are gradients of some Hamiltonian functions, $\Lambda_i^{\sharp}$ $(i = 1, 2, \ldots)$are called implectic operators. Let M be a smooth manifold. We can think of Poisson bivector Λas a skew form on the cotangent bundle. Then we can express the implectic operator by vector bundle map associated to Λ, that is,

$$\Lambda^{\sharp} : T^*M \longrightarrow TM.$$

Actually the name “implectic” is derived from inverse “symplectic”. It is also called cosymplectic operators.

2.2. Lie–Poisson structure

The Lie–Poisson bracket on the dual of Lie algebra $\mathfrak{g}^*$ is given by

$$\{f, g\}(\mu) = \langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\rangle, \tag{11}$$

where $\frac{\delta f}{\delta \mu}$ is the unique element of $\mathfrak{g}$ such that

$$\langle \nu, \frac{\delta f}{\delta \mu}\rangle = \mathrm{d}f(\mu).\nu$$

holds for all $\nu \in \mathfrak{g}^*$.

Lemma 2.1. *The Hamiltonian vector field on $\mathfrak{g}^*$ corresponding to a Hamiltonian function f, computed with respect to the Lie–Poisson structure is given by*

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = ad^*_{\mathrm{d}f}\mu \tag{12}$$

Proof. It follows from the following identities:

$$i_{X_f}\,\mathrm{d}g|_\mu = L_{X_f} g|_\mu = \{f, g\}_{LP}(\mu) = \langle [\mathrm{d}g, \mathrm{d}f], \mu\rangle = \langle \mathrm{d}g, ad^*_{\mathrm{d}f}\mu\rangle.$$

This implies that $X_f = ad^*_{\mathrm{d}f}\mu$. Thus the Hamiltonian equation $\frac{\mathrm{d}\mu}{\mathrm{d}t} = X_f$ yields our result. □

Let I be an inertia operator:

$$I : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and then $\mu \in \mathfrak{g}^*$ evolve by

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = (I^{-1}\mu)\mu, \tag{13}$$

where right hand side denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This equation is called the Euler–Poincaré equation.

Definition 2.2. The Euler–Poincaré equation on $\mathfrak{g}^*$ corresponding to the Hamiltonian $H(\mu) = \frac{1}{2}\langle I^{-1}\mu, \mu\rangle$ is given by

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = -ad^*_{I^{-1}\mu}\mu.$$

It characterizes an evolution of a point $\mu \in \mathfrak{g}^*$.

Proposition 2.3. *Let ΩG be infinite dimensional Lie group equipped with a right invariant metric. A curve $t \longrightarrow c(t)$ in ΩG is a geodesic of this metric iff $u(t) = d_{c_t} R_{c_t^{-1}} \dot{c}(t)$ satisfies:*

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -ad^*_{u(t)}u(t). \tag{14}$$

2.2.1. Euler–Poincaré flow and KdV

Now let us consider the prototype example in the infinite dimensional setting. Let us apply this to the KdV equation:

$$u_t = u_{xxx} + 6uu_x. \tag{15}$$

The interpretation of the KdV equation as an Euler–Poincaré flow on the space of Hill's operator M is based on the fact that (10) can be written:

$$u_t = \Lambda^{\sharp}\frac{\delta H}{\delta u}, \tag{16}$$

where

$$\Lambda^{\sharp} = \tfrac{1}{2}\partial^3 + 2u\partial + u' \tag{17}$$

and $H = \langle u(x), u(x)\rangle = \int_{S^1} u^2\,\mathrm{d}x$.

2.3. Kirillov superalgebra

The first characteristic special property of superalgebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$.

A Lie superalgebra is Z_2-graded algebra:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

An element v of $\mathfrak{g}_0$(resp. $\mathfrak{g}_1$) is said to be even (resp. odd). The supercommutator of a pair of elements $v, w \in \mathfrak{g}$is defined by

$$[v, w] = vw - (-1)^{\tilde{v}\tilde{w}}wv,$$

where $\tilde{v}$and $\tilde{w}$are the gradings of $v$and w, respectively. It also satisfies super Jacobi identity:

$$(-1)^{\tilde{v}\tilde{u}}[v, [w, u]] + (-1)^{\tilde{v}\tilde{w}}[w, [u, v]] + (-1)^{\tilde{w}\tilde{u}}[u, [v, w]].$$

Let Ωbe the cotangent bundle of S^1. Let us denote $\Omega^{\pm 1/2}$be the square root of the tangent and cotangent bundle of S^1, respectively.

Let us define

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where we denote $\mathfrak{g}_0 \equiv Vect(S^1)$and $\mathfrak{g}_1 \equiv \Omega^{-1/2}(S^1)$. $\mathfrak{g}$forms a super Lie algebra on $S^{1,1}$[13,14] and $\mathfrak{g}_1$is the super-partner of $\mathfrak{g}_0$. This is asserted since $\mathfrak{g}_1$is the $\mathfrak{g}_0$module and it is compatible with the structure of $\mathfrak{g}_0$module and satisfies $\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$. A typical element of $\mathfrak{g}$would be

$$f(x)\frac{\mathrm{d}}{\mathrm{d}x} + \xi(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}},$$

and the super Lie bracket is given by

$$[(f_1, \xi_1), (f_2, \xi_2)] = ([f_1, f_2] + \xi_1\xi_2, \{f_1, \xi_2\} + \{\xi_1, f_2\}).$$

Definition 2.4. A superprojective vector field is a pair $(f\mathrm{d}/\mathrm{d}x, \psi\sqrt{\mathrm{d}/\mathrm{d}x})$which satisfies:

$$f''' + 4f'u + 2fu' = 0.$$

and

$$\xi'' + u\xi = 0.$$

In this realization $\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x} \oplus \xi(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}\right)$, i.e. $(f(x), \xi(x))$forms a super Lie algebra. $(f(x), \xi(x))$satisfies:

$$f(x + 2\pi) = f(x), \qquad \xi(x + 2\pi) = \pm\xi(x).$$

When it is in the '+' sector, it is called Ramond sector super Lie algebra and '–' sector is known as Neveu–Schwarz sector.

The cocycle may be extended to this superalgebra via

$$c(\xi_1, \xi_2) = \int_{S^1} \xi_1' \xi_2' \, \mathrm{d}x. \tag{18}$$

3. Coadjoint orbit associated to superalgebra and fisher type flows

Let us concentrate on the coadjoint action of the odd (or Fermionic) part of the Ramond and Neveu–Schwarz superalgebras.

Proposition 3.1. *Let* $\hat{\xi} = \left(\xi(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}, a\right)$ *and* $\hat{u} = (u(x)\mathrm{d}x^2, c)$. *Then the coadjoint action of* $\hat{\xi}$ *on* $\hat{u}(x)$ *yields*:

$$ad^*_{\hat{\xi}} \hat{u}(x) = \left(-c\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x)\right)\xi. \tag{19}$$

Sketch of Proof. It is clear

$$\left[\left(\xi(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}, a\right), \left(\eta(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}, b\right)\right] := \xi(x)\eta(x)\frac{\mathrm{d}}{\mathrm{d}x}, \int_{S^1} \xi'\eta' \, \mathrm{d}x.$$

Thus,

$$\langle ad^*_{\hat{\xi}(x)\sqrt{\mathrm{d}/\mathrm{d}x}} \hat{u}(x)\mathrm{d}x^2, \hat{\eta}(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}\rangle = \langle \hat{u}(x)\mathrm{d}x^2, \left[\hat{\xi}(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}, \hat{\eta}(x)\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}\right]\rangle$$

$$= \langle (u(x)\mathrm{d}x^2, c), \xi\eta\frac{\mathrm{d}}{\mathrm{d}x}, \int_{S^1} \xi'\eta' \, \mathrm{d}x\rangle = \langle (-c\xi'' + u(x)\xi, 0), \hat{\eta}\sqrt{\frac{\mathrm{d}}{\mathrm{d}x}}\rangle \quad \square$$

3.1. Euler–Poincaré form of Fisher equation

The Hamiltonian operator corresponding to "Fermionic" part of the Kirillov's superalgebra is

$$\mathcal{O}_{\mathrm{Fer}} = -c\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x). \tag{20}$$

If we carry out the above calculation when the dual of the vector field is $u + \Lambda$, for $\Lambda \in \mathbf{R}$. Thus we obtain

Lemma 3.2.

$$\mathcal{O}_{\mathrm{Fisher}} = ad^*_{\hat{\xi}}(\hat{u} + \Lambda) = \left(-c\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) + \Lambda\right)\xi. \tag{21}$$

We will study the Euler–Poincaré flow associated to this operator.

Proposition 3.3. *The Euler–Poincaré flow on the odd part of the superalgebra yields the pseudospherical type equation* [23,24]:

$$u_t + u^2 - cu_{xx} = 0. \tag{22}$$

Sketch of Proof. Using Euler–Poincaré's theorem:

$$u_t = -\mathcal{O}_{\mathrm{Fer}} \frac{\delta H}{\delta u(x)},$$

we obtain the above result. □

Let us consider now $\mathcal{O}_{\mathrm{Fisher}}$Hamiltonian operator. Thus,

$$u_t = -\mathcal{O}_{\mathrm{Fisher}} \frac{\delta H}{\delta u(x)}$$

yields Fisher equation.

Remark 3.4. Let us confine to hyperplane $c = 1$, then Eq. (22) is a reduction of the Fitzhugh–Nagumo equation:

$$u_t - u_{xx} + u(u-1)(u-\gamma) = 0. \tag{23}$$

The Fitzhugh–Nagumo equation is a simplification of the Hodgkin–Huxley model devised in 1952.

Remark 3.5. The Lie–Poisson structure associated to symmetric operator $\mathcal{O}_{\mathrm{Fisher}} = \left(-c\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) + \Lambda\right)$is symmetric.

Corollary 3.6. *The following reduction of the Fitzhugh–Nagumo equation*:

$$u_t + u(u-1)(u-\gamma) = 0 \tag{24}$$

associated to the hyperplane $c = 0$*is the Euler–Poincaré flow on the dual space of the odd part of Kirillov's algebra for*

$$H = \frac{1}{4}u^4 - \frac{\gamma+1}{3}u^3 + \frac{\gamma}{2}u^2.$$

4. Leibniz bracket and extended Fitzhugh–Nagumo system

A Leibniz algebra is a vector space $\mathfrak{g}$equipped with a binary operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$which satisfies the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad x, y, z \in \mathfrak{g}.$$

Lie algebras are trivial examples of Leibniz algebras, those with antisymmetric bracket.

Definition 4.1. Let M be a smooth manifold and let $C^\infty(M)$be the ring of smooth functions on it. A Leibniz bracket on M is a bilinear map $[\cdot,\cdot]_{\text{Leib}} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$that is a derivation in each entry:

$$[fg, h]_{\text{Leib}} = [f, h]_{\text{Leib}} g + f[g, h]_{\text{Leib}}, \qquad [f, gh]_{\text{Leib}} = g[f, h]_{\text{Leib}} + h[f, g]_{\text{Leib}}.$$

Let Πbe the Poisson bivector. The Schouten bracket $[\Pi, \Pi]_{\text{Schouten}}$is determined by

$$\tfrac{1}{2}\langle [\Pi, \Pi]_{\text{Schouten}}, \, \mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h\rangle = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\},$$

so that $[\Pi, \Pi]_{\text{Schouten}} = 0$if and only if the Lie algebra satisfies Jacobi identity.

Definition 4.2. An almost Poisson manifold is a pair $(M, [\cdot,\cdot]_{\text{Leib}})$and the bracket $[\cdot,\cdot]_{\text{Leib}}$does not satisfy Jacobi identity. An almost Poisson structure on M will be Poisson manifold if its Jacobiator $\mathcal{J} : C^\infty(M) \times C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$given by

$$\mathcal{J}(f, g, h) = [[f, g], h] + [[g, h], f] + [[h, f], g]$$

vanishes.

Let us search for an infinite dimensional analog of the Leibniz bracket. Let construct the Lie–Poisson bracket associated to the KdV operator and the operator associated to the action of the square root of the vector field on the dual of the Virasoro algebra.

Lemma 4.3. *The Lie–Poisson bracket associated to the action of Kirillov's superalgebra on the dual*:

$$\{f, g\}_{\text{Leib}} := \int_{S^1} \frac{\delta f}{\delta u} \mathcal{O}_{eFN} \frac{\delta g}{\delta u} \mathrm{d}x, \tag{25}$$

where

$$\mathcal{O}_{eFN} = \partial^3 + \partial u + u\partial + \mu(\partial^2 + u).$$

Proof. It is easy to see that the KdV part induces the skew part and the super part yields symmetric part of the bracket, that is,

$$\mathcal{O}_{eFN}^{\text{skew}} = \partial^3 + \partial u + u\partial, \qquad \mathcal{O}_{eFN}^{\text{sym}} = \partial^2 + u.$$

So the diffusion part is coming from the symmetric part of $\mathcal{O}_{eFN}$. □

Let us study the dynamics involved with the above bracket. Thus using Eq. (25) we obtain the extended FN equation:

$$u_t = \{u, H\}_{\text{Leib}} = u_{xxx} + \mu u_{xx} + 3uu_x + \mu u^2,$$

for $H = \frac{1}{2}\int_{S^1} u^2 \, \mathrm{d}x$.

Remark 4.4. *Physically* Jacobiator associated to the extended Fitzhugh–Nagumo system only implies the existence of source and sink. This is very natural for dissipative systems. Jacobiator naturally vanishes in the case of integrable systems.

5. Burgers–Huxley equation

In this section we will show that the Burgers–Huxley equation fit into the category of the Euler–Poincaré flows.

5.1. Burgers Flow

Let us consider a first-order differential operator:

$$\Delta_1 = \frac{\mathrm{d}}{\mathrm{d}x} + u(x). \tag{26}$$

This $\Delta_1$satisfies:

$$\Delta_1 = \frac{\mathrm{d}}{\mathrm{d}x} + u(x) : \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_{-1/2}. \tag{27}$$

Definition 5.1. The $Vect(S^1)$- action on $\Delta_1$is defined by the commutator with the Lie derivative:

$$[\mathcal{L}_{f(x)\mathrm{d}/\mathrm{d}x}, \Delta_1] := \mathcal{L}^{-1/2}_{f(x)\mathrm{d}/\mathrm{d}x} \circ \Delta_1 - \Delta_1 \circ \mathcal{L}^{1/2}_{f(x)\mathrm{d}/\mathrm{d}x}. \tag{28}$$

The result of this action is a scalar operator, i.e. the operator of multiplication by a function.

Lemma 5.2.

$$[\mathcal{L}_{f(x)\mathrm{d}/\mathrm{d}x}, \Delta_1] = \tfrac{1}{2} f''(x) + u f_x(x) + u_x f(x). \tag{29}$$

Proof. By direct computation. □

Hence the Hamiltonian operator (after rescaling) is

$$\mathcal{O}_{\text{Burgers}} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x. \tag{30}$$

5.1.1. Geometrical interpretation

By Lazutkin and Penkratova [16], this dual space can be identified with the space of Hill's operator:

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x), \tag{31}$$

where u is a periodic potential: $u(x + 2\pi) = u(x) \in C^{\infty}(\mathbb{R})$. The Hill's operator maps:

$$\Delta : \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{-\frac{3}{2}}. \tag{32}$$

The action of $Vect(S^1)$on the space of Hill's operator the Hill's operator is equivalent to the relation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}v(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}v(x)\right), \tag{33}$$

where

$$u = \frac{1}{2}\left(v_x - \frac{1}{2}v^2\right),$$

giving the formal factorization of the Hill's operator.

Geometrically this can be realized as

$$\mathcal{F}_{1/2} \xrightarrow{\Delta_1} \mathcal{F}_{-1/2} \xrightarrow{\Delta^1} \mathcal{F}_{-3/2},$$

where $\Delta = \Delta^1\Delta_1 = (\partial - \frac{1}{2}v)(\partial + \frac{1}{2}v)$. This is compatible with $\Delta : \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_{-3/2}$.

Thus the Burgers operator can be related to the Hamiltonian operator obtained from the action of $Vect(S^1)$on the square root of dual.

5.2. Computation of Burgers–Huxley equation

Let us consider the following Hamiltonian operator:

$$\mathcal{O}_{BH} := \mathcal{O}_{\text{Burgers}} + \beta\mathcal{O}_1 \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x + \beta(u + \Lambda), \tag{34}$$

where $\Lambda \in \mathbf{R}$.

Once again, the operator $\mathcal{O}_{BH}$is a mixture of both skew and symmetric operators given as

$$\mathcal{O}_{BH}^{\text{skew}} = 2u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x, \quad \text{and} \quad \mathcal{O}_{BH}^{\text{sym}} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(u + \Lambda). \tag{35}$$

Remark 5.3. There is a duality exist between the two halves of the $\mathcal{O}_{BH}$. the second part of the Hamiltonian operator $\mathcal{O}_{BH}$obtained from the action of the square root of the vector field on the dual of the vector field. On the other hand the first part is obtained from the action of the vector field on the dual of the square root of the vector field.

Therefore, the Euler–Poincaré flows associated to $\mathcal{O}_{BH}$becomes:

$$u_t = \mathcal{O}_{BH}\frac{\delta H}{\delta u} \equiv \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x + \beta(u + \Lambda)\right)u = u_{xx} + 4uu_x + \beta(u^2 + \Lambda u).$$

This equation reduces to the Burgers equation when $\beta = 0$. The Lie–Poisson bracket associated to $\mathcal{O}_{BH}$is given by

$$\{f, g\}_{\text{Leib}} = \{f, g\}_{\text{skew}} + \{\{f, g\}\}_{\text{sym}} = \int \frac{\delta f}{\delta u}\mathcal{O}_{BH}^{\text{skew}}\frac{\delta g}{\delta u} + \int \frac{\delta f}{\delta u}\mathcal{O}_{BH}^{\text{sym}}\frac{\delta g}{\delta u},$$

where $\mathcal{O}_{BH}^{\text{skew}}$and $\mathcal{O}_{BH}^{\text{sym}}$are given in (35).

5.3. Extended Burgers–Huxley equation

Let us consider second Hamiltonian given by the Sobolev (2,1)-norm:

$$H_2 : W^{2,1}(\mathbf{R}) \longrightarrow \mathbf{R}, \qquad H_2 = \int_{S^1} (u(x)^2 + \alpha^2 u_x(x)^2).$$

Let us consider the dual of the Lie algebra of $\mathfrak{g}^*$with a Poisson structure given by the "frozen" Lie–Poisson structure. In otherwords, we fix some point $\mu_0 \in \mathfrak{g}^*$and define a Poisson structure given by

$$\{f, g\}_{\text{Frozen}}(\mu) := \langle[\mathrm{d}f(\mu), \mathrm{d}g(\mu)], \mu_0\rangle.$$

Let us decompose the Hamiltonian operator $\mathcal{O}_{BH}$into u-independent and -dependent part:

$$\mathcal{O}_{BH} = \mathcal{O}_{BH}^0 + \mathcal{O}_{BH}^u,$$

respectively. Thus, $\mathcal{O}_{BH}^0 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Lambda$.

Definition 5.4. The "Frozen" Leibniz or symmetric bracket associated to $\mathcal{O}^0_{BH}$is defined as

$$\{\{f, g\}\}_{\text{Frozen}} = \int \frac{\delta f}{\delta u}\mathcal{O}^0_{BH}\frac{\delta g}{\delta u}, \tag{36}$$

where $\mathcal{O}^0_{BH} = \frac{d^2}{dx^2} + \Lambda$.

Proposition 5.5. *The following Euler–Poincaré flows associated to the action of Vect*(S^1)*the space of first-order differential operators*:

$$u_t = \mathcal{O}^0_{BH}\frac{\delta H_2}{\delta u} + \lambda \mathcal{O}^u_{BH}\frac{\delta H_1}{\delta u}$$

yields the extended Burgers–Huxley equation:

$$u_t = -\lambda u_{xxxx} + (1 - \Lambda)u_{xx} + 4uu_x + \beta(u^2 + \Lambda u), \tag{37}$$

for Hamiltonian $H_1 = \int_{S^1} u^2\, dx$*and* $H_2 = \int_{S^1}(u(x)^2 + \alpha^2 u_x(x)^2)$.

Proof. By direct computation. □

5.4. KdV–Burgers equation

The KdV and Burgers equations appear quite naturally in plasma physics. The dispersion of the ion-acoustic wave balances the positive nonlinearity to form solitons. It is the physics behind the KdV equation. For a wave which possesses a positive nonlinearity like the ion-acoustic wave, damping like Landau damping or collisional damping is needed to form shock waves which results in the Burgers equation. Recently Nakamura et. al. [21] observed shock waves in an unmagnetized dusty plasma. They asserted that the development of the shock is due to KdV–Burgers equation. The KdV–Burgers for their purpose is

$$w_t + ww_x + \beta w_{xxx} + \mu w_{xx} = 0, \tag{38}$$

where $\beta$and $\mu$are constants.

The *Vect*(S^1)action the pencil of operators $\Delta^{\lambda,\mu} := \lambda\Delta_2 + \mu\Delta_1$is given by

$$[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta^{\lambda,\mu}] = \lambda[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_2] + \mu[\mathcal{L}_{f(x)\frac{d}{dx}}, \Delta_1], \tag{39}$$

where

$$\Delta_2 = \frac{d^2}{dx^2} + u(x) \quad \text{and} \quad \Delta_1 = \frac{d}{dx} + u(x).$$

The pencil of Poisson structures corresponding to (39) is given by

$$\mathcal{O}_{\lambda,\mu} = \lambda\left(\tfrac{1}{2}\partial^3 + \partial u + u\partial\right) + \mu(\partial^2 + \partial u). \tag{40}$$

The correspondong Leibniz–Poisson bracket is given as

$$\{f, g\}_{\text{Leib}} = \int_{S^1}\frac{\delta f}{\delta u}\mathcal{O}_{\lambda,\mu}\frac{\delta g}{\delta u}dx,$$

where

$$\{f, g\}_{\text{skew}} = \int_{S^1}\frac{\delta f}{\delta u}\lambda\left(\frac{1}{2}\partial^3 + \partial u + u\partial + \mu\partial u\right)\frac{\delta g}{\delta u}dx$$

and

$$\{\{f, g\}\}_{\rm sym} = \int_{S^1} \frac{\delta f}{\delta u}(\mu\partial^2)\frac{\delta g}{\delta u}\,{\rm d}x.$$

Proposition 5.6. *The Euler–Poincaré flow* $u_t = -O_{\lambda,\mu}\frac{\delta H}{\delta u}$ *is given by*

$$u_t + \tfrac{1}{2}\lambda u_{xxx} + \mu u_{xx} + (3\lambda + 2\mu)uu_x = 0. \tag{41}$$

6. Hamiltonian form, Leibniz operator, Lie pseudogroup and final outlook

Let us review quickly the KdV phenomena:

$$u_t = u_{xxx} + 3uu_x.$$

It has the Hamiltonian form:

$$u_t = (\partial^3 + u\partial + u\partial)\frac{\delta H}{\delta u}, \qquad H(u) = \int_{S^1} u^2\,{\rm d}x. \tag{42}$$

Let us follow Wilson's [26,27] argument. Let $\mathcal{U}$ be the linear space of all smooth periodic functions $u(x)$. To establish a link with Hamiltonian equation we have to regard the skew operator $\mathcal{O}_{\rm KdV}$ occuring in KdV equation as a contravariant rank 2 tensor on the manifold $\mathcal{U}$. Thus, we can use $\mathcal{O}_{\rm KdV}$ to define Lie–Poisson bracket of two functions f and g, that is,

$$\{f, g\} = \int_{S^1} \frac{\delta f}{\delta u}\mathcal{O}_{\rm KdV}\frac{\delta g}{\delta u}\,{\rm d}x.$$

If the operator $\mathcal{O}_{\rm KdV}$ is invertible, then the inverse operator $\Omega_{\rm KdV} = (\mathcal{O}_{\rm KdV})^{-1}$ would define a 2-form on the space $\mathcal{U}$. In fact, Wilson showed that $\mathcal{O}_{\rm KdV}$ can be inverted on some slightly larger function space to yield symplectic 2-form connected to the Schwarzian KdV equation.

Let us consider the Hamiltonian form of Leibniz operator $\mathcal{O}_{\rm Leib}$, we set:

$$u_t = \mathcal{O}_{\rm Leib}\frac{\delta H}{\delta u}.$$

We use $\mathcal{O}_{\rm Leib}$ to define the Leibniz–Poisson bracket. The important property of the operator $\mathcal{O}_{\rm Leib}$ is that the Poisson bracket does not satisfy Jacobi identity. It satisfies Jacobi identity upto a closed 3-form, Jacobiator.

We now come to an important philosophical question. If we can invert $\mathcal{O}_{\rm Leib}$ then the inverse operator $\Omega = (\mathcal{O}_{\rm Leib})^{-1}$ would then define 2-form on $\mathcal{U}$. The fact that the Leibniz–Poisson bracket satisfies the Jacobiator $\mathcal{J}(f, g, h)$ would then correspond to 2-form with conformal vector field.

The vector field X^r is said to be conformal with parameter $r \in \mathbf{R}$ if

$$L_{X^r}\Omega = r\Omega, \tag{43}$$

and the diffeomorphism ϕ^r is conformal if $(\phi^r)^*\Omega = r\Omega$. The diffeomorphisms ϕ^r form the pseudogroup $Diff^r_\Omega$.

If, in addition, $H^1(U) = 0$, then the conformal vector is given by

$$\{X^r = X_H + rZ : H \in C^\infty(U)\},$$

where X_H is a Hamiltonian vector field and Z is defined by

$$i_Z\Omega = -\theta. \tag{44}$$

Using the homotopy formula for Lie derivatives we obtain that $X^r$is conformal on (U, Ω)if and only if $\mathrm{d}i_{X^r}\Omega = r\Omega$. Thus, Ωmust be exact $\Omega = -\mathrm{d}\theta$for some $\theta$for all values of $r \neq 0$. Therefore, Eq. (43) boils down to

$$L_{X^r}\Omega = L_Z\Omega = \mathrm{d}i_{X^r}\Omega = -r\mathrm{d}\theta. \tag{45}$$

Let us construct Lie–Poisson bracket connected to conformal vector X^r. From the definition of X^rwe have

$$\mathrm{d}fX^r = \mathrm{d}f(X_H + Z) = \{f, H\} - i_Z\Omega(X_f) = \{f, H\} + r\theta(X_f).$$

Let $M = \mathbf{R}^{2n}$be a phase space, then the conformal Hamiltonian system associated to $\theta = p\,\mathrm{d}q$and $\Omega = \mathrm{d}p \wedge \mathrm{d}q$is given by

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} - rp. \tag{46}$$

In this paper we study various nonlinear-reaction diffusion systems. The dynamics of these classes of systems are governed by Leibniz mechanics. These are similar to conformal Hamiltonian systems and underlying symmetry group is Lie pseudogroup. What is interesting here that all these equations are related to the super action of Kirillov's local algebra which in turn related to the Virasoro algebra. These are all Euler–Poincaré flows on the dual of the Kirillov's superalgebra. In partcular, these are obtained from the super action on bosonic to fermionic part and vice-versa. One class of systems are connected to the EP flow associated to the action of the square root of the vector field (odd part of the Kirillov's algebra) on the dual of vector field, and another class of systems are related to the action of $Vect(S^1)$on the "square root" of the dual of $Vect(S^1)$.

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